Almost Gorenstein rings

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Introduction -History of Commutative Algebra-

- In the end of the 19th century, commutative ring theory was originally established by D. Hilbert, proved *Hilbert's Basis Theorem*.
- E. Noether played a central role of the developments of the theory of commutative algebra. At the middle of the 20th century, the notion of homological method was innovated into commutative ring theory by many researchers, say M. Auslander, D. A. Buchsbaum, D. Rees, D. G. Northcott, J.-P. Serre and others.
- J.-P. Serre finally proved an innovative result which claims that every localization of a regular local ring is again regular.
- Since then, and up to the present day, commutative ring theory has been developed dramatically by investigating the theory of Cohen-Macaulay rings and modules.

Introduction -My research interest-

Main interest ⋯ Classification of (local) rings in terms of homological algebra

Hierarchy of local rings

Regular ⇒ Complete Intersection ⇒ Gorenstein ⇒ Cohen–Macaulay ⇒ Buchsbaum ⇒ generalized Cohen–Macaulay (FLC)

Introduction -Cohen–Macaulay rings-

- A Noetherian ring R satisfies the unmixedness theorem

$\overset{def}{\iff}$ $\forall$ ideal I of R generated by $\mathrm{ht}_R\, I$ elements is unmixed, namely, $\mathrm{Ass}_R\, R/I = \mathrm{Min}_R\, R/I$.

- (F. S. Macaulay) Polynomial ring over a field satisfies the unmixedness theorem.
- (I. S. Cohen) Regular local ring satisfies the unmixedness theorem.

Definition 1.1

Let R be a Noetherian local ring. Then

$$R \text{ is a Cohen–Macaulay ring} \overset{def}{\iff} R \text{ satisfies unmixedness theorem} \iff \dim R = \operatorname{depth} R$$

Introduction -Gorenstein rings-

Definition 1.2

Let R be a Noetherian local ring. Then

$$R \text{ is a Gorenstein ring} \quad \overset{def}{\Longleftrightarrow} \quad \mathrm{id}_R\, R < \infty$$
$$\Longleftrightarrow \quad R \text{ is Cohen–Macaulay}, \quad R \cong \mathrm{K}_R$$

- Gorenstein rings $\Longrightarrow$ Cohen–Macaulay rings
- Gorenstein rings have a beautiful symmetry.

Introduction

Example 1.3 (Determinantal rings)

Let $S = k[X_{ij} \mid 1 \leq i \leq m, 1 \leq j \leq n\,]$ $(2 \leq m \leq n)$ be the polynomial ring over a field k and put

$$R = S/\mathrm{I}_t(X)$$

where $2 \leq t \leq m$, $\mathrm{I}_t(X)$ is the ideal of S generated by $t \times t$-minors of $X = [X_{ij}]$.

Then

$$R \text{ is a Gorenstein ring} \quad \Longleftrightarrow \quad m = n.$$

Example 1.4 (Numerical semigroup rings)

- $0 < a_1 < a_2 < \cdots < a_\ell \in \mathbb{Z}$ s.t. $\gcd(a_1, a_2, \ldots, a_\ell) = 1$
- $H = \langle a_1, a_2, \ldots, a_\ell \rangle = \{ \sum_{i=1}^{\ell} c_i a_i \mid 0 \le c_i \in \mathbb{Z} \}$
- $R = k[[H]] := k[[t^{a_1}, t^{a_2}, \ldots, t^{a_\ell}]] \subseteq V := k[[t]]$
- $\mathfrak{m} = (t^{a_1}, t^{a_2}, \ldots, t^{a_\ell})$
- $c = \mathrm{c}(H) := \min\{n \in \mathbb{Z} \mid m \in H, \text{ if } m \in \mathbb{Z}, \ m \ge n\} < \infty$
- $\mathrm{K}_R = \sum_{n \in \mathbb{Z} \setminus H} R t^{a-n}$, where $a = c - 1$

Then

$$R \text{ is a Gorenstein ring} \iff H \text{ is symmetric.}$$

$$\begin{aligned} H \text{ is symmetric} \quad &\overset{\text{def}}{\iff} \quad \forall n \in \mathbb{Z}, \ [\ n \in H \iff c - 1 - n \notin H\ ] \\ &\iff \quad \sharp\{n \in H \mid n < c\} = \sharp(\mathbb{N} \setminus H) \\ &\iff \quad \sharp(\mathbb{N} \setminus H) = \frac{c}{2} \end{aligned}$$

Example 1.5

(1) $k[[t^4, t^5, t^6]]$: Gorenstein ring

(2) $k[[t^3, t^5, t^7]]$: not Gorenstein ring

(3) $k[[t^3, t^7, t^8]]$: not Gorenstein ring

$H = \langle 4, 5, 6 \rangle$

0	1	2	3
4	5	6	7
8	9	10	11
12		$\cdots$	

$\mathrm{K}_R = R$

$H = \langle 3, 5, 7 \rangle$

0	1	2
3	4	5
6	7	8
9	$\cdots$	

$\mathrm{K}_R = R + Rt^2$

$\mathfrak{m}\,\mathrm{K}_R \subseteq R$

$H = \langle 3, 7, 8 \rangle$

0	1	2
3	4	5
6	7	8
9	$\cdots$	

$\mathrm{K}_R = R + Rt$

$\mathfrak{m}\,\mathrm{K}_R \nsubseteq R$

Question 1.6

Why are there so many Cohen-Macaulay rings which are not Gorenstein?

Aim of this research

Find a new class of Cohen-Macaulay rings which may not be Gorenstein, but sufficiently good next to Gorenstein rings.

$\cdots$ Almost Gorenstein rings

Introduction

History of almost Gorenstein rings

- [Barucci-Fröberg, 1997]

  $\cdots$ one-dimensional analytically unramified local rings

- [Goto-Matsuoka-Phuong, 2013]

  $\cdots$ one-dimensional Cohen-Macaulay local rings

- [Goto-Takahashi-T, 2015]

  $\cdots$ higher-dimensional Cohen-Macaulay local/graded rings

Almost Gorenstein local rings

Setting 2.1

- $(R, \mathfrak{m})$ a Cohen-Macaulay local ring with $d = \dim R$
- $|R/\mathfrak{m}| = \infty$
- $\exists$ K_R the canonical module of R

Definition 2.2 (Goto-Takahashi-T, 2015)

We say that R is *an almost Gorenstein local ring*, if $\exists$ an exact sequence

$$0 \to R \to \mathrm{K}_R \to C \to 0$$

of R-modules such that $\mu_R(C) = \mathrm{e}^0_{\mathfrak{m}}(C)$.

Look at an exact sequence

$$0 \to R \to \mathrm{K}_R \to C \to 0$$

of R-modules. If $C \neq (0)$, then C is Cohen-Macaulay and $\dim_R C = d-1$.

Set $\overline{R} = R/[(0) :_R C]$.

Then $\exists\ f_1, f_2, \dots, f_{d-1} \in \mathfrak{m}$ s.t. $(f_1, f_2, \dots, f_{d-1})\overline{R}$ forms a minimal reduction of $\overline{\mathfrak{m}} = \mathfrak{m}\overline{R}$. Therefore

$$\mathrm{e}^0_{\mathfrak{m}}(C) = \mathrm{e}^0_{\overline{\mathfrak{m}}}(C) = \ell_R(C/(f_1, f_2, \dots, f_{d-1})C) \geq \ell_R(C/\mathfrak{m}C) = \mu_R(C).$$

Thus

$$\mu_R(C) = \mathrm{e}^0_{\mathfrak{m}}(C) \Longleftrightarrow \mathfrak{m}C = (f_1, f_2, \dots, f_{d-1})C.$$

Hence C is a *maximally generated maximal Cohen-Macaulay $\overline{R}$-module* in the sense of B. Ulrich, which is called *an Ulrich R-module*.

Definition 2.3

We say that R is *an almost Gorenstein local ring*, if $\exists$ an exact sequence

$$0 \to R \to \mathrm{K}_R \to C \to 0$$

of R-modules such that either $C = (0)$ or $C \neq (0)$ and C is an Ulrich R-module.

Theorem 2.4 (Goto-Matsuoka-Phuong)

Suppose that $d = 1$ *and* $R \subseteq \mathrm{K}_R \subseteq \overline{R}$. *Then*

$$R \text{ is an almost Gorenstein ring} \quad \Longleftrightarrow \quad \mathfrak{m}K_R \subseteq R.$$

Example 2.5

(1) $k[[t^4, t^5, t^6]]$: Gorenstein ring

(2) $k[[t^3, t^5, t^7]]$: almost Gorenstein ring $(\mathfrak{m}\,\mathrm{K}_R \subseteq R)$

(3) $k[[t^3, t^7, t^8]]$: not almost Gorenstein ring $(\mathfrak{m}\,\mathrm{K}_R \not\subseteq R)$

Moreover, if $H = \langle 3, a, b \rangle$ $(3 < a < b,\ \gcd(3, a, b) = 1)$, then

$$R : \text{almost Gorenstein ring} \quad \Longleftrightarrow \quad b = 2a - 3.$$

Theorem 2.6 (NZD characterization)

(1) *If R is a non-Gorenstein almost Gorenstein local ring of dimension $d > 1$, then so is $R/(f)$ for genaral NZD $f \in \mathfrak{m} \setminus \mathfrak{m}^2$.*

(2) *Let $f \in \mathfrak{m}$ be a NZD on R. If $R/(f)$ is an almost Gorenstein local ring, then so is R. When this is the case, $f \notin \mathfrak{m}^2$, if R is not Gorenstein.*

Corollary 2.7

Suppose that $d > 0$. If $R/(f)$ is an almost Gorenstein local ring for every NZD $f \in \mathfrak{m}$, then R is Gorenstein.

Example 2.8 (T, 2017)

Let $S = k[[X_{ij} \mid 1 \le i \le m, 1 \le j \le n\,]]$ $(2 \le m \le n)$ be the formal power series ring over an infinite field k and put

$$R = S/\mathrm{I}_t(X)$$

where $2 \le t \le m$, $X = [X_{ij}]$.

Then

$$R \text{ is an almost Gorenstein local ring} \quad \Longleftrightarrow \quad m = n, \text{ or } m \ne n,\ t = m = 2$$

Theorem 2.9

Let $(S, \mathfrak{n})$ be a Noetherian local ring, $\varphi : R \to S$ a flat local homomorphism. Suppose that $S/\mathfrak{m}S$ is a RLR. Then TFAE.

(1) *R is an almost Gorenstein local ring.*

(2) *S is an almost Gorenstein local ring.*

Therefore

- R is almost Gorenstein $\iff$ $R[[X_1, X_2, \ldots, X_n]]$ is almost Gorenstein.
- R is almost Gorenstein $\iff$ $\widehat{R}$ is almost Gorenstein.

The following is a generalization of the result of Goto-Matsuoka-Phuong.

Theorem 2.10

Suppose that $d > 0$. Let $\mathfrak{p} \in \operatorname{Spec} R$ and assume that $R/\mathfrak{p}$ is a RLR of dimension $d-1$. Then TFAE.

(1) *$A = R \ltimes \mathfrak{p}$ is an almost Gorenstein local ring.*

(2) *R is an almost Gorenstein local ring.*

Example 2.11

Let k be an infinite field. We consider

$$A = k[[X, Y, Z, U, V, W]]/I$$

where

$$I = (X^3 - Z^2, Y^2 - ZX) + (U, V, W)^2 + (YU - XV, ZU - XW, ZU - YV, ZV - YW, X^2U - ZW).$$

Then

$$A \cong k[[t^4, t^5, t^6]] \ltimes (t^4, t^5, t^6)$$

and hence A is an almost Gorenstein local ring.

Theorem 2.12

Let $(R, \mathfrak{m})$ *be a Cohen-Macaulay complete local ring with* $\dim R = 1$ *and assume that* $R/\mathfrak{m}$ *is algebraically closed of characteristic* 0.
Suppose that R *has finite CM representation type. Then* R *is an almost Gorenstein local ring.*

Theorem 2.13 (Goto)

Suppose that R is a non-Gorenstein almost Gorenstein local ring with $\dim R \geq 1$*. Let M be a finitely generated R-module. If*

$$\operatorname{Ext}_R^i(M, R) = (0)$$

for $\forall i \gg 0$*, then* $\operatorname{pd}_R M < \infty$*.*

Corollary 2.14

Suppose that R is an almost Gorenstein local ring with $\dim R \geq 1$*. If R is not a Gorenstein ring, then R is G-regular in the sense of [4], i.e.*

$$\operatorname{Gdim}_R M = \operatorname{pd}_R M$$

for every finitely generated R-module M.

Semi-Gorenstein local rings

In this section we maintain Setting 2.1.

Definition 3.1

We say that R is *a semi-Gorenstein local ring*, if R is an almost Gorenstein local ring which possesses an exact sequence

$$0 \to R \to \mathrm{K}_R \to C \to 0$$

such that either $C = (0)$, or C is an Ulrich R-module and $C = \oplus_{i=1}^{\ell} C_i$ for some cyclic R-submodule C_i of C.

Therefore, if $C \neq (0)$, then

$$C_i \cong R/\mathfrak{p}_i \quad \text{for } \exists\ \mathfrak{p}_i \in \operatorname{Spec} R$$

such that $R/\mathfrak{p}_i$ is a RLR of dimension $d-1$.

Notice that

- almost Gorenstein local ring with $\dim R = 1$
- almost Gorenstein local ring with $\mathrm{r}(R) \leq 2$

are semi-Gorenstein.

Proposition 3.2

Let R be a semi-Gorenstein local ring. Then $R_\mathfrak{p}$ is semi-Gorenstein for $\forall \mathfrak{p} \in \operatorname{Spec} R$.

Therefore, if $C \neq (0)$, then

$$C_i \cong R/\mathfrak{p}_i \quad \text{for } \exists\, \mathfrak{p}_i \in \operatorname{Spec} R$$

such that $R/\mathfrak{p}_i$ is a RLR of dimension $d-1$.

Notice that

- almost Gorenstein local ring with $\dim R = 1$
- almost Gorenstein local ring with $\mathrm{r}(R) \leq 2$

are semi-Gorenstein.

Proposition 3.2

Let R be a semi-Gorenstein local ring. Then $R_{\mathfrak{p}}$ is semi-Gorenstein for $\forall \mathfrak{p} \in \operatorname{Spec} R$.

Theorem 3.3

Let $(S, \mathfrak{n})$ be a RLR, $\mathfrak{a} \subsetneq S$ an ideal of S with $n = \mathrm{ht}_S\, \mathfrak{a}$. Let $R = S/\mathfrak{a}$. Then TFAE.

(1) *R is a semi-Gorenstein local ring, but not Gorenstein.*

(2) *R is Cohen-Macaulay, $n \geq 2$, $r = \mathrm{r}(R) \geq 2$, and R has a minimal S-free resolution of the form:*

$$0 \to F_n = S^r \overset{\mathbb{M}}{\to} F_{n-1} = S^q \to F_{n-2} \to \cdots \to F_1 \to F_0 = S \to R \to 0$$

where

$${}^t\mathbb{M} = \begin{pmatrix} y_{21} y_{22} \cdots y_{2\ell} & y_{31} y_{32} \cdots y_{3\ell} & \cdots & y_{r1} y_{r2} \cdots y_{r\ell} & z_1 z_2 \cdots z_m \\ x_{21} x_{22} \cdots x_{2\ell} & 0 & 0 & 0 & 0 \\ 0 & x_{31} x_{32} \cdots x_{3\ell} & 0 & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & x_{r1} x_{r2} \cdots x_{r\ell} & 0 \end{pmatrix},$$

$\ell = n + 1$, $q \geq (r-1)\ell$, $m = q - (r-1)\ell$, and $x_{i1}, x_{i2}, \ldots, x_{i\ell}$ is a part of a regular system of parameters of S for $2 \leq \forall i \leq r$.

When this is the case

$$\mathfrak{a} = (z_1, z_2, \ldots, z_m) + \sum_{i=2}^{r} \mathrm{I}_2 \begin{pmatrix} y_{i1} & y_{i2} & \cdots & y_{i\ell} \\ x_{i1} & y_{i2} & \cdots & x_{i\ell} \end{pmatrix}.$$

Example 3.4

Let $\varphi : S = k[[X, Y, Z, W]] \longrightarrow R = k[[t^5, t^6, t^7, t^9]]$ be the k-algebra map defined by

$$\varphi(X) = t^5,\ \varphi(Y) = t^6,\ \varphi(Z) = t^7 \text{ and } \varphi(W) = t^9.$$

Then

$$0 \to S^2 \xrightarrow{\mathbb{M}} S^6 \to S^5 \to S \to R \to 0,$$

where

$${}^t\mathbb{M} = \begin{pmatrix} W & X^2 & XY & YZ & Y^2-XZ & Z^2-XW \\ X & Y & Z & W & 0 & 0 \end{pmatrix}.$$

Hence R is semi-Gorenstein with $\mathrm{r}(R) = 2$ and

$$\operatorname{Ker}\varphi = (Y^2 - XZ, Z^2 - XW) + \mathrm{I}_2 \begin{pmatrix} W & X^2 & XY & YZ \\ X & Y & Z & W \end{pmatrix}.$$

Almost Gorenstein graded rings

Setting 4.1

- $R = \bigoplus_{n \geq 0} R_n$ a Cohen-Macaulay graded ring with $d = \dim R$
- $(R_0, \mathfrak{m})$ a Noetherian local ring
- $|R_0/\mathfrak{m}| = \infty$
- $\exists$ K_R the graded canonical module of R
- $M = \mathfrak{m}R + R_+$
- $a = \mathrm{a}(R) := -\min\{n \in \mathbb{Z} \mid [\mathrm{K}_R]_n \neq (0)\}$

Definition 4.2

We say that R is *an almost Gorenstein graded ring*, if $\exists$ an exact sequence

$$0 \to R \to \mathrm{K}_R(-a) \to C \to 0$$

of graded R-modules such that $\mu_R(C) = \mathrm{e}^0_M(C)$.

Notice that

- R is an almost Gorenstein graded ring
  $\Longrightarrow$ R_M is an almost Gorenstein local ring.

Theorem 4.3

Let $R = k[R_1]$ *be a Cohen-Macaulay homogeneous ring with* $d = \dim R \geq 1$*. Suppose that* $|k| = \infty$ *and* R *is not a Gorenstein ring. Then TFAE.*

(1) R *is an almost Gorenstein graded ring and level.*

(2) $\mathrm{Q}(R)$ *is a Gorenstein ring and* $\mathrm{a}(R) = 1 - d$*.*

Example 4.4

Let $S = k[X_{ij} \mid 1 \leq i \leq m, 1 \leq j \leq n\ ]$ $(2 \leq m \leq n)$ be the polynomial ring over an infinite field k and put

$$R = S/\mathrm{I}_t(X)$$

where $2 \leq t \leq m$, $X = [X_{ij}]$.

Then R is an almost Gorenstein graded ring if and only if either $m = n$, or $m \neq n$ and $t = m = 2$.

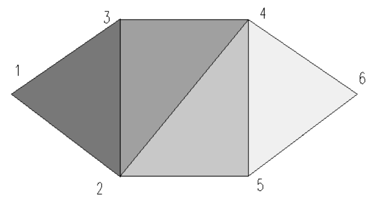
Example 4.5

Let $R = k[X_1, X_2, \ldots, X_d]$ $(d \geq 1)$ be a polynomial ring over an infinite field k. Let $n \geq 1$ be an integer.

- $R^{(n)} = k[R_n]$ is an almost Gorenstein graded ring, if $d \leq 2$.
- Suppose that $d \geq 3$. Then $R^{(n)}$ is an almost Gorenstein graded ring if and only if either $n \mid d$, or $d = 3$ and $n = 2$.

Example 4.6

Look at the simplicial complex Δ :



Then $R = k[\Delta]$ is an almost Gorenstein graded ring of dimension 3, provided $|k| = \infty$.

Two-dimensional rational singularities

Setting 5.1

- $(R, \mathfrak{m})$ a Cohen-Macaulay local ring with $d = \dim R$
- $|R/\mathfrak{m}| = \infty$
- $\exists$ K_R the canonical module of R
- $v(R) = \mu_R(\mathfrak{m})$, $\mathrm{e}(R) = \mathrm{e}^0_{\mathfrak{m}}(R)$
- $G = \mathrm{gr}_{\mathfrak{m}}(R) = \bigoplus_{n \geq 0} \mathfrak{m}^n/\mathfrak{m}^{n+1}$

Theorem 5.2

(1) *Suppose that R is an almost Gorenstein local ring and $v(R) = \mathrm{e}(R) + d - 1$. Then G is an almost Gorenstein graded ring and level.*

(2) *Suppose that G is an almost Gorenstein graded ring and level. Then R is an almost Gorenstein local ring.*

Corollary 5.3

Suppose that $v(R) = \mathrm{e}(R) + d - 1$. *Then TFAE.*

1. R *is an almost Gorenstein local ring.*
2. G *is an almost Gorenstein graded ring.*
3. $\mathrm{Q}(G)$ *is a Gorenstein ring.*

Corollary 5.4

Suppose that $v(R) = \mathrm{e}(R) + d - 1$ *and* R *is a normal ring. If* $\mathfrak{m}$ *is a normal ideal, then* R *is an almost Gorenstein local ring.*

Corollary 5.5

Every two-dimensional rational singularity is an almost Gorenstein local ring.

Corollary 5.6

Every two-dimensional Cohen-Macaulay complete local ring R of finite CM representation type is an almost Gorenstein local ring, provided R contains a field of characteristic 0.

Thank you so much for your attention.

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